THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5220 Complex Analysis and its Applications 2016-2017
Suggested Solution to mid-term

1 Let $g(z) = f(z) - 1$. Since f is analytic on $|z| \leq R$, g is also analytic on $|z| \leq R$. By maximum principle, $|g(z)|$ attains its maximum at some point z_0 in the boundary. As a result, we have $g(z) \leq g(z_0) < 1$ for all $|z| \leq R$. By triangle inequality, for $|z| < R$, we have

$$|f(z)| = |g(z) + 1| \geq 1 - |g(z)| \geq 1 - |g(z_0)| > 1 - 1 = 0$$

Hence $f(z) \neq 0$ for $|z| < R$.

2 Since $f(z)$ is an entire function, it has a Talyor's series expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$, where

$$a_k = \int_{|z|=r} \frac{f(s)}{s^{k+1}} ds, \ r \geq R$$

As a result, we have

$$|a_k| = \left| \int_{|z|=r} \frac{f(s)}{s^{k+1}} ds \right| \leq 2\pi r \frac{Mr^n}{r^{k+1}} = \frac{2\pi M}{r^{k-n}}$$

In particular, for $k > n$, since r can be arbitrarily large,

$$|a_k| \leq \frac{2\pi M}{r^{k-n}} \xrightarrow{r \to \infty} 0$$

Hence $f(z) = \sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{n} a_k z^k$ must be a polynomial of degree at most n.

3 (a) Note that $z^4 + 6z + 5 = (z^2 + 1)(z^2 + 5)$. By triangle inequality, for $|z| = R > \sqrt{5}$, we have

$$|z^4 + 6z^2 + 5| = |(z^2 + 1)(z^2 + 5)| \geq (R^2 - 1)(R^2 - 5) \text{ and } |z + 1| \leq |z| + 1 = R + 1$$

Hence

$$\left|\int_{C_R} \frac{z - 1}{z^4 + 6z^2 + 5} dz\right| \leq \frac{R + 1}{(R^2 - 1)(R^2 - 5)} 2\pi R = \frac{2\pi R}{(R - 1)(R^2 - 5)}$$

(b) Since $|\pm i\sqrt{5}| = \sqrt{5} > 2$, inside the contour $|z| = 2$ the integrand is not analytic only at the point $z = \pm i$. By deformation of path, we can find $\epsilon > 0$ small enough such that

$$\int_{C_R} \frac{z - 1}{z^4 + 6z^2 + 5} dz = \int_{|z-i|=\epsilon} \frac{z - 1}{z^4 + 6z^2 + 5} dz + \int_{|z+i|=\epsilon} \frac{z - 1}{z^4 + 6z^2 + 5} dz$$

By Cauchy's Integral formula, we have

$$\int_{|z-i|=\epsilon} \frac{z - 1}{z^4 + 6z^2 + 5} dz = \int_{|z-i|=\epsilon} \frac{(z - 1)/[(z + i)(z^2 + 5)]}{z - i} dz = 2\pi i \frac{(i - 1)}{(i + i)(i^2 + 5)} = \frac{\pi(i - 1)}{4}$$

$$\int_{|z+i|=\epsilon} \frac{z - 1}{z^4 + 6z^2 + 5} dz = \int_{|z+i|=\epsilon} \frac{(z - 1)/[(z - i)(z^2 + 5)]}{z + i} dz = 2\pi i \frac{(-i - 1)}{(-i - i)((-i)^2 + 5)} = \frac{\pi(i + 1)}{4}$$

Hence

$$\int_{C_R} \frac{z - 1}{z^4 + 6z^2 + 5} dz = \frac{\pi(i - 1)}{4} + \frac{\pi(i + 1)}{4} = \frac{\pi i}{2}$$

4 When $1+z^4=-r$ for some $r\geq 0$, we have $z^4=-r-1=(r+1)e^{i\pi}$. Hence $z=(r+1)^{\frac{1}{4}}e^{i(\frac{\pi}{4}+\frac{2k\pi}{4})}$ for $k=0,1,2,3$. As a result, the maximum domain of definition is given by

$$\mathbb{C}\backslash\{re^{i\theta}|r\geq 1,\theta=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4}\text{ or }\frac{7\pi}{4}\}$$

5 By Laurent series expansion, we have $e^{z+\frac{1}{z}}=\sum_{n=-\infty}^{\infty}a_nz^n$, where $a_n=\frac{1}{2\pi i}\int_{|z|=1}\frac{e^{z+\frac{1}{z}}}{z^{n+1}}dz$. In particular, for $n\neq 0$,

$$\begin{aligned}
a_n =&\frac{1}{2\pi i}\int_{|z|=1}\frac{e^{z+\frac{1}{z}}}{z^{n+1}}dz\\
=&\frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{e^{(e^{i\theta}+e^{-i\theta})}}{(e^{i\theta})^{n+1}}de^{i\theta}\\
=&\frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{e^{2\cos\theta}}{(e^{i\theta})^{n+1}}ie^{i\theta}d\theta\\
=&\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{2\cos\theta}e^{-in\theta}d\theta\\
=&\frac{1}{2\pi}(\int_{-\pi}^{\pi}e^{2\cos\theta}\cos(n\theta)d\theta-i\int_{-\pi}^{\pi}e^{2\cos\theta}\sin(n\theta)d\theta)\\
=&\frac{1}{\pi}\int_{0}^{\pi}e^{2\cos\theta}\cos(n\theta)d\theta,
\end{aligned}$$

where in the last equality we use the facts that sine and cosine functions are odd and even respectively. This gives the desired result.

6 Let L be the length of the contour C. Given $z\in\Omega$, define $r=\min_{w\in C}|z-w|$ to be the distance between the point z and the contour C. Then for any Δz with $|\Delta z|<r/2$, we have $|s-(z+\Delta z)|\geq|s-z|-|\Delta z|\geq r/2$. Note that

$$\frac{1}{\Delta z}\left(\int_C\frac{f(s)}{s-(z+\Delta z)}ds-\int_C\frac{f(s)}{s-z}ds\right)=\int_C\frac{f(s)}{(s-z)(s-(z+\Delta z))}ds$$

Since $f(s)$ is a continuous function, $M=\max_{s\in\bar{\Omega}}f(s)$ exists. From this we have

$$\begin{aligned}
&\left|\frac{1}{\Delta z}\left(\int_C\frac{f(s)}{s-(z+\Delta z)}ds-\int_C\frac{f(s)}{s-z}ds\right)-\int_C\frac{f(s)}{(s-z)^2}ds\right|\\
=&\left|\int_C\frac{f(s)}{(s-z)(s-(z+\Delta z))}ds-\int_C\frac{f(s)}{(s-z)^2}ds\right|\\
=&\left|\int_C\frac{f(s)\Delta z}{(s-z)^2(s-(z+\Delta z))}ds\right|\\
\leq&L\times\frac{M}{r^2(r/2)}\Delta z\xrightarrow{\Delta z\to 0}0
\end{aligned}$$

This gives the desired result.